Lecture 23

Tuesday, 28 April 2026 9:25 AM

Solving the LP:

$\boxed{\text{CONF - LP}}$

$$\max \frac{1}{n} \sum_{i \in N} \sum_{S \subseteq G} y_{i,S} \ln v_i(S) \qquad\qquad \min \sum_{i \in N} \alpha_i + \sum_{j \in G} \beta_j$$

$$\text{s.t.} \quad \forall i, \ \sum_{S \subseteq G} y_{i,S} \leq 1 \qquad\qquad \forall i, S, \ \alpha_i + \sum_{j \in S} \beta_j \geq \ln v_i(S)/n$$

$$\forall j \in G \quad \sum_{i,S : j \in S} y_{i,S} \leq 1 \qquad\qquad \alpha_i, \beta_j \geq 0$$

$$\forall i, S \quad y_{i,S} \geq 0$$

Thm: Given $\varepsilon > 0$, can in time polynomial in $|I|$ & $1/\varepsilon$ obtain a a solution $(y_{i,S})$ to CONF - LP so that $(y_{i,S}) \in \mathbb{Q}$, has support at most poly $(|I|, 1/\varepsilon)$, and has value $\geq$ OPT $- \ln(1+\varepsilon)$

Assume for now we know OPT.
Let's focus on solving the dual for now. Need an (approx) separation oracle.

Approx sep oracle: Given $(\alpha_i)_{i \in N}$, $(\beta_j)_{j \in S}$, find $S \subseteq G$, $i \in N$ s.t.

$$\alpha_i + \sum_{j \in S} \beta_j < \frac{\ln v_i(S)}{n} \quad \text{—— } (*)$$

Claim: Given α, β s.t. for some i, S, $(*)$ holds, can efficiently find i, S s.t. $\alpha_i + \sum_{j \in S} \beta_j < \frac{1}{n} \ln\left((1+2\varepsilon) v_i(S)\right)$

Proof: Let i^*, S^* be s.t. $(*)$ does not hold. Let $v^* = \max_{j \in S^*} v_{i^* j}$.

Suppose we guess i^*, v^* (iterate over all choices — nm)
Discard all goods j: $v_{i^* j} > v^*$. Let G' be remaining set of goods
$\forall i, j$, $\bar{v}_{i^* j} = v_{i^* j}$ round down to integer multiple of $\frac{\varepsilon}{2m} v^*$

$$= \left\lfloor \frac{v_{i^* j}}{\frac{\varepsilon}{2m} v^*} \right\rfloor$$

Note that there are $\frac{2m}{\varepsilon}$ choices for values, & hence $\frac{2m^2}{\varepsilon}$ choices

for $\bar{v}_{i^*}(S^*)$
Now we 'guess' $\bar{v}_{i^*}(S^*) =: \lambda^*$
Note that $\bar{v}_{i^*}(S^*) \leq v_{i^*}(S^*)$
& $\bar{v}_{i^*}(S^*) \geq v_{i^*}(S^*) - m \cdot \frac{\varepsilon}{2m} v^* \geq v_{i^*}\left(1 - \frac{\varepsilon}{2}\right)$
So if $\alpha_{i^*} + \sum_{j \in S^*} \beta_j < \frac{1}{n} \ln v_{i^*}(S^*)$

then $\alpha_{i^*} + \sum_{j \in S^*} \beta_j < \frac{1}{n} \ln (1+\varepsilon)\, \bar{v}_{i^*}(S^*) = \frac{1}{n} \ln (1+\varepsilon) \lambda^*$

or $\sum_{j \in S^*} \beta_j < \underbrace{\frac{1}{n} \ln (1+\varepsilon) \lambda^* - \alpha_{i^*}}_{\text{known (guessed)}}$

Thus we want to find $S \subseteq G'$ s.t. (i) $\sum_{j \in S} \beta_j < \frac{1}{n} \ln(1+\varepsilon) \lambda^* - \alpha_{i^*}$

(ii) $\bar{v}_{i^*}(S) \geq \lambda^*$

This corresponds to a knapsack instance w/ G' goods,
β_j is "size" of good j, $\bar{v}_{ij}$ is value.
But there are only $\frac{2m}{\varepsilon}$ diff. values for $\bar{v}_{i^* j}$, so can

solve this by DP. $\blacksquare$

So why is an approx oracle good enough?
Run ellipsoid. For any $(\alpha_i)_{i \in N}$, $(\beta_j)_{j \in G}$ returned, check if

(i) $\sum_{i \in N} \alpha_i + \sum_{j \in G} \beta_j \leq \text{OPT} - \varepsilon$

(ii) if claim returns i, S s.t. $\alpha_i + \sum_{j \in S} \beta_j < \frac{1}{n} \ln (1+2\varepsilon) v_i(S)$,
return constraint $\alpha_i + \sum_{j \in S} \beta_j \geq \frac{1}{n} \ln v_i(S)$

If both condns check, then $\exists\, \alpha, \beta$ s.t.

$$\sum_{i \in N} \alpha_i + \sum_j \beta_j \leq \text{OPT} - \varepsilon$$

$$\forall i, S \quad \alpha_i + \sum_{j \in S} \beta_j \geq \frac{1}{n} \ln v_i(S)$$

which cannot be.
Hence, the ellipsoid must return that this is infeasible.
In particular, the LP

$$\sum_i \alpha_i + \sum_j \beta_j \leq \text{OPT} - \varepsilon$$

$$\sum_{j \in S} \beta_j + \alpha_i \geq \frac{1}{n} \ln (1+2\varepsilon) v_i(S) \qquad \forall (i,S) \text{ returned to ellipsoid}$$

is infeasible.
Thus, the LP

$$\max \frac{1}{n} \sum_i \sum_{S \subseteq G} y_{i,S} \ln (1+2\varepsilon) v_i(S)$$

$$\text{s.t.} \quad \forall i \quad \sum_S y_{i,S} \leq 1$$

$$\forall j \quad \sum_{i,S : j \in S} y_{i,S} \leq 1$$

$$y_{i,S} \geq 0,$$

supported on (i,S) queried by the ellipsoid algo, has optimal value $\geq$ OPT $- \varepsilon$.

There is the question of $(1+2\varepsilon)$ in the objective, but will let you work out how to handle that.

Last bit: how to guess OPT?
Fix. ε. Suppose we have lower & upper bounds L & H on OPT. so that $H/L \leq m$
Fix $\varepsilon' = \varepsilon/H$.
Run algo for all $o \in \{L, L(1+\varepsilon), L(1+\varepsilon)^2, \ldots, H\}$ pick best soln.